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An immunization-hedging investment strategy for a future portfolio of corporate bonds

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by

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I. INTRODUCTION

The focus of this study is interest rate risk reduction. It is particularly relevant now because of the recent volatility of interest rates. Increased interest rate volatility leads to increased uncertainty with regard to the return for an investment. There are tools available to the portfolio manager which will enable him/her to decrease the uncertainty caused by rapidly changing market values and reinvestment rates. This study will incorporate into an investment strategy two of the most widely used tools: hedging and immunization.

Managing a portfolio consisting of fixed-income securities has become an extremely difficult occupation. A "portfolio" is simply a group of fixed-income securities. When interest rates are subject to significant change, the portfolio becomes exposed to two types of risk. First, there is a direct relationship between the income received from the reinvestment of the coupon payments and changes in interest rates. If rates are increasing, the coupons may be reinvested at this higher rate. The second type of risk involves the inverse relationship between changes in interest rates and the market value of the portfolio. The market price of a fixed income security varies inversely with a change in interest rates. A manager's success is measured by how effectively he deals with these two types of risk.

The purpose of this study is to define and test a method which allows the fixed-income-securities manager to protect the total portfolio return from unexpected changes in interest rates. This method is called an immunization-hedging procedure for a future investment in high grade corporate bonds. U.S. Treasury Bond futures contracts will be used for hedging during the time period from when the manager is informed of the investment (t0) to when he/she actually receives the funds (t1) to make the purchase. Then the immunization procedure will be incorporated for the length of a predetermined holding period (tn - t1). This method should provide maximum protection from adverse movements in interest rates for the future value of the portfolio. It should also provide the manager with a close approximation of the realized return over the life of the investment, or in terms of price, the future value of the portfolio.

In 1938, Macaulay (25) derived a measure of a bond's price sensitivity to a change in the discount factor (1+r) and called this measure duration. It has not received much attention in the academic journals until recently because of the extreme volatility of interest rates. The concept of duration has experienced a rebirth. Duration is a weighted average measure of time, where the weights are expressed in present value terms. Although there are many ways to calculate duration, Macaulay's formula is presented below. In mathematical form

$$D = \frac{\sum_{t=1}^{n} \frac{C_t \cdot t}{(1+r)^t} + \frac{A \cdot n}{(1+r)^n}}{\sum_{t=1}^{n} \frac{C_t}{(1+r)^t} + \frac{A}{(1+r)^n}} \tag{1}$$

where D is Macaulay's measure of duration, C_t is the cash flow from the bond in period t (i.e., the coupon payment), r is the yield to maturity, t is the number of years to the cash flow payment, n is the number of years to maturity of the bond, and A is the face value of the bond. Note that the denominator is the price of the asset. The numerator is equal to the present value of the t-th period's cash receipt multiplied by the number of years to payment.

Duration matching is normally the technique used to immunize a portfolio of fixed-income securities. When a portfolio is arranged so that its duration is equal to the length of the investor's holding period, the portfolio is said to be immunized. Immunization assures the manager of receiving at least the return promised by the term structure of the interest rates at the time the investment is made. The minimum return will not decrease, and may increase, even if interest rates change during the holding period. The return promised is realized because the increase in income received from the reinvestment of the coupons is at least as large as the decline in the market value of the portfolio, assuming interest rates increase. If interest rates fall, the opposite will be true.

Prior to the actual purchase of a portfolio of fixed-income securities, interest rates may decrease resulting in a higher market price. A manager needs a different method to insulate the initial purchase price of the portfolio from adverse movements in interest rates. Hedging will be used for this purpose. An example will help to clarify this point. Suppose on December 1, 1983, a manager learns that in three months he/she will be given $10 million to invest in AAA corporate bonds. If interest rates fall between now and March 1, 1984, the $10 million will not be able to purchase as many securities as it could have in December. A purchase of T-bond futures contracts in December could have decreased the chance of incurring an opportunity loss caused by the falling interest rates. The gain from the futures purchase should have offset most, if not all, of the opportunity loss. As the previous example showed, hedging involves taking a position in the futures market which is opposite to the position in the spot market.

In this study, hedging with the T-bond futures contract is different from the type of hedging farmers normally use. When an investor is planning to purchase a portfolio of fixed-income securities, he/she will buy T-bond futures contracts. A farmer who finds the current futures price for his/her corn crop attractive is able to lock-in an attractive selling price by selling corn futures contracts. The former type of hedge is called an anticipatory hedge whereas the farmer's hedge is called a cash hedge. The investor is "anticipating" a purchase of fixed-income securities whereas the farmer already owns the asset underlying the futures contract.

The goal of hedging is typically not profit maximization but risk reduction. The risk, as stated earlier, appears to have increased over the past few years due to the uncertainty of movements in interest rates. While it is true that the hedger cannot be assured of perfect price correlation between the cash and futures market, hedging is not as risky as outright price speculation. Hedging is, and will continue to be, the main vehicle by which market participants are able to transfer risk.

The introduction has highlighted the main aspects of this thesis. Chapter II will describe the financial futures market in general and hedging in more detail. The concepts of duration and immunization are discussed in Chapter III. A literature review is presented in Chapter IV. The investment model, and also a simulation of this model, is the subject of Chapter V. The results will be discussed in the final chapter.

II. FINANCIAL FUTURES AND HEDGING

A. Futures Markets in General

This chapter will begin by discussing the futures market in general. It will describe the users of the market, types of traders, and other unique aspects of the futures market. A detailed discussion of hedging will be presented along with examples to help clarify esoteric concepts. The specifics of the U.S. Treasury bond futures contract will also be presented. The factors affecting the hedge ratio are then analyzed followed by a comparison of four methods used to calculate the optimal number of contracts to trade.

Before examining the details of the T-bond futures contract and the hedging process, a brief explanation of the futures market in general is in order. Initially, futures trading was a method for farmers (wholesalers) dealing in grains, to hedge the selling (purchase) price to be received (paid). Over the years it has evolved into an enormous market dealing not only in grains, but also in metals, livestock, meat, petroleum, food, fiber, oilseeds, wood, and financials. The futures market is regulated by the Commodity Futures Trading Commision (CFTC). Formed in 1974 by the passage of the Commodity Futures Trading Commission Act, the CFTC's objectives are 1) "to foster competition in the market place" and 2) "to protect market participants from fraud, deceit and abusive practices" (Powers (27), 1981, p. 261).

There are two basic types of market participants in the futures industry. Speculators are those who willingly accept a risky position in return for a chance at making a profit. They normally do not use or own the cash commodity which underlies the futures contract they trade. Another type of speculator is called a spreader. He/she will take a position in two different contracts of the same commodity, the same contract but different exchanges, or two different contracts. The spreader may be short the March T-bond contract and long the June contract, long the March contract at the Chicago Board of Trade and short the March contract at the MidAmerica Commodity Exchange, or be long the March T-bond contract and short the March GNMA contract. Hedgers, on the other hand, are risk-averse individuals whose objective is to reduce their exposure to risk. Hedgers decrease their risk exposure by "trading the basis" rather than individual contract prices or differences in contract prices as the spreader does. Unlike the speculator, the hedger typically owns and has a use for the cash commodity. An arbitrageur technically is someone who buys something cheap and sells it dear making a profit without assuming any risk. In practice, a true arbitrage rarely exists.

Each commodity exchange is required to maintain a clearinghouse. The purpose of the clearinghouse is to match each day's buy and sell orders. It becomes a party to every transaction. The buyer does not trade directly with the seller, he must deal directly with the clearinghouse, which in turn will contract with the seller. This serves to guarantee delivery and also helps to maintain an orderly market. The clearinghouse collects its members' losses, caused by price changes during the day's trading, and pays the members who have a gain on their position. In essence, it acts as a collection and payment agency by settling its members' accounts after each day of trading as each individual account is marked-to-market. A firm, which is a member of the exchange, debits or credits each client's account and the clearinghouse debits or credits each member's account.

The notion of delivery is another aspect unique to the futures market. Even though only a very small fraction of the contracts traded are ever delivered, the price of a contract is based upon the idea that delivery may occur. Delivery is actually a three-day process involving the selling clearing member, the clearinghouse, and the buying clearing member. Financials, traded at the Chicago Board of Trade, may be delivered during many months of the year but March, June, September, and December are the most common delivery months. In addition, delivery may occur only on certain days during these months and these days differ for each contract. T-bonds, traded at the Chicago Board of Trade, may be delivered on any business day during the delivery month or on the last two business days of the previous month for the remaining days of the month. T-bonds stop trading on the eighth business day before the end of the month. The settlement price that prevails on this day will determine the invoice amount. A problem arises because the futures price does not change after this time, but cash bond prices will. Whether or not the seller will deliver depends upon how much cash prices change during the last eight business days. Also, the cheapest-to-deliver bond may change during the end of the delivery month. Cash bond prices must be monitored even though the T-bond futures contract has expired.

Financial publications which list daily price activity for futures contracts report open interest in the contracts. Open interest is simply the number of open transactions. A transaction is open if it has not been offset or fulfilled by delivery. It is necessary for each open transaction to have a buyer and a seller, but only one side is counted when calculating open interest.

Open interest can increase when new purchases are offset by new sales. "New" refers to buyers and sellers just entering the market or taking on new positions in the market. Open interest decreases when old sellers purchase from old buyers or when old buyers sell to old sellers. Old buyers (sellers) have outstanding long (short) positions. It should be pointed out that it normally takes a high volume of trading to change open interest substantially.

Anyone planning to hedge in the futures market will have to become familiar with the concept of basis. Basis is defined to be the difference between the cash price and the futures price. Normally, the long-term interest rate futures price will be less than the cash price, implying a positive basis. This occurs because the typical shape for the yield curve is upward-sloping which means funds can be borrowed today more cheaply than the return available on a longer-term investment. The futures market will react by pricing the contracts furthest from delivery lower than the nearby contracts. A strengthing of the basis means that it is becoming more positive. The trading range for the basis is usually much narrower than the range for the cash instrument or the futures contract. This is why trading the basis is relatively safe when compared to trading individual cash or futures contract prices.

The basis will fluctuate within a small range during most of the life of the hedge. It tends toward zero, but usually is not equal to zero, in the delivery month. The reason this occurs is because the cash price and the futures contract price converge as the contract matures since holding a futures contract with only a few days before maturity is essentially the same as a spot position.

The shape of the yield curve will help to determine whether the basis is positive or negative. An upward-sloping yield curve means that the price of a longer-term bond is lower than that of a shorter-term one with equal coupons (i.e., the basis is positive). A negatively-sloped yield curve implies a negative basis.

A margin account is created when a position is taken. The size of the margin account is determined by the contract traded. The initial margin for a T-bond futures contract is \$1,000-\$1,250 and the maintenance margin is \$1,000. Margins are usually higher for a speculator (\$1,250) than for a hedger (\$1,000). At the end of each trading day, each account's gain or loss for the day is calculated. The accounts which show a gain are credited and the ones showing a loss are debited (i.e., marked to market). Funds are subtracted from the loser's margin account and transferred to the gainer's margin account.

A gain may be withdrawn from the margin account but a loss, if it causes the account to fall below its maintenance level, will have to be covered. This implies that the cash flow of the hedger may be volatile during the life of the hedge. Also, if one's hedge consistently produced a gain, the funds could be withdrawn and reinvested causing the actual gain on the hedge to be larger than the difference between the selling and buying (or buying and selling) price. Obviously, an account which continually exhibits a loss will create a cash drain for the trader.

Commissions for a futures transaction are extremely low when compared to the value of the asset underlying the contract. The charge will normally be in the neighborhood of $50 per round-trip transaction but commissions as low as $20 have been encountered by the author. The size of the commission varies with the number and frequency of contracts traded.

Before going into a detailed discussion concerning hedging, some possible users of the financial futures market will be described. Any institution which deals heavily in the money market or the bond market will be a candidate to use the financial futures market. These institutions may decide to hedge the purchase price of their portfolios from falling interest rates by buying futures contracts. Figure 1 shows how hedging will help to protect an investor from incurring an opportunity loss on a future purchase of corporate bonds if interest rates decrease.

Suppose on June 1 an investor decides to purchase $1 million of ten percent corporate bonds August 1. The current price of the bond is 828.44. A purchase of ten T-bond futures contracts (par value $1 million) on June 1 for $68,312.50 each should offset any increase in the cash T-bond price between now and August 1. By August 1 the cash bond is

Cash market		Futures market	
June 1:	Decide to purchase $1 million 10 percent corporate bonds due 2004. Price equals 82-27.	June 1:	Purchase 10 T-bond contracts at 68-10.
August 1:	Purchase bonds for 92-31 to yield 8.75 percent.	August 1:	Sell 10 T-bond contracts for 78-14.
Opportunity loss:	$101,250	Futures gain:	$101,250

Figure 1. Anticipatory long hedge[a]

[a]This is an example of a perfect hedge (loss equals gain) because the basis is the same in August (14-17) as it was in June (14-17).

selling for $929.69 and the futures contract price has risen to $78,437.50. The opportunity loss of $101,250 created by an increase in the cash bond price is exactly offset by the $101,250 gain on the ten T-bond futures contracts.

This is an example of a perfect anticipatory long hedge, and it shows why hedging should be considered. As pointed out in the introduction, interest rates today are very volatile which may cause large fluctuations in the market value of a portfolio of fixed-income securities. This variance in the value of the portfolio can be reduced by correctly formulating a hedging strategy. The institution referred to in the previous paragraph may be an insurance company anticipating a purchase of corporate bonds or a commercial bank which is planning to purchase a large portfolio of government bonds. Other possible users include investment bankers, bond dealers, pension fund managers and corporate treasurers. Anyone wishing to transfer interest rate risk should hedge.

When hedging, a portfolio manager will take a position in the futures market which is equal to his/her expected cash position (anticipatory hedge) or if he/she currently has a cash position, the futures position will be opposite to this cash position (cash hedge). Since the anticipatory hedge has already been illustrated, a brief description of a cash hedge will be presented. For instance, if a manager is planning to sell part of a currently held bond portfolio in the near future (long position), he/she will sell (short position) T-bond futures contracts today to protect the proceeds of the sale. The proceeds will decrease in value if interest rates should rise before the transaction is completed but the futures position should create a profit. In theory, the former's loss should equal the latter's gain.

B. The U.S. Treasury Bond Contract

This study will focus on using the U.S. Treasury Bond contract as a vehicle for hedging during the time period prior to the actual purchase of the portfolio. One reason the T-bond contract is so favorable is that it is the most actively traded financial futures contract. This means that it is the most liquid. Liquidity is desirable because it provides the user with an opportunity to easily change or cancel his position if it becomes necessary. Another reason is that the closer the asset underlying the futures contract is to the asset being hedged, the closer are their price movements over time. This helps to keep the basis stable. Because of this, the manager is provided with the greatest opportunity to limit the risk exposure of the portfolio caused by fluctuating interest rates.

The T-bond futures contract has a face value of \$100,000 and is based upon a 15-year, eight percent coupon. Any U.S. Treasury Bond may be delivered in fulfillment of the contract as long as it has at least 15 years to maturity from the delivery date if not callable. If the bond is callable, it must have at least 15 years remaining to call from the delivery date. Bond contract prices are quoted in percentage points of par. For example, a bond contract which is quoted at 91-02 means that it sells for 91 and two 32nd percentage points of par of the basic deliverable bond. Since each 32nd is worth \$31.25, the contract will sell for \$91,062.50. Minimum price fluctuations are one 32nd of a point with the daily limit move set at two points (\$2,000). The hedger must post a \$1,000 initial margin and the maintenance is also \$1,000. If the balance in the maintenance margin falls below \$1,000, the hedger will have to deposit enough to bring the balance up to \$1,000 again. Margins are essentially performance bonds which help to guarantee the financial integrity of both parties.

The formula for calculating the invoice amount is given by

$$\text{Invoice} = (\underset{\text{price}}{\text{settlement}} \cdot \$100{,}000) \cdot \underset{\text{factor}}{\text{conversion}} + \underset{\text{interest}}{\text{accrued}}. \quad (2)$$

The conversion factor is a number which, when multiplied by the settlement price, will be the price at which a delivered bond will yield eight percent. The purpose of the conversion factor is to price the eligible Treasury securities whose characteristics (coupon and term to maturity) do not match the specifications of the futures contract. Many factors are necessary because a large number of Treasury issues qualify for delivery at a point in time. Obviously, these issues have various coupons and maturity dates. A bond which has a coupon greater than eight percent will have a conversion factor greater than one. The opposite is true for a bond which has a coupon less than eight percent. It also takes into consideration the time to maturity, or the time to call, of the issue. The settlement price in equation (2) is given in decimal form. Accrued interest is found by multiplying the daily interest times the number of days from the beginning of the current six-momth interest payment period until the delivery date.

C. Determining the Hedge Ratio

Since this study is concerned with an anticipatory long hedge, all discussion pertaining to hedging will concern itself with only this type of hedge. One of the problems confronting the manager lies in determining the optimal number of contracts to trade. More commonly known as the hedge ratio, its calculation will be the major determinant in the effectiveness of the hedge. Factors which affect the size of the hedge ratio include: the par value and market value of the cash instrument versus the face value and market value of the futures contract, the maturity of the asset underlying the futures contract and the maturity of the cash instrument, the size of the coupon for the cash instrument and the asset underlying the futures contract, differences in the risk structure of interest rates, and the shape of the yield curve.

The par or market value of the cash instrument and the futures contract will have an effect upon the hedge ratio. For instance, a T-bond futures contract calls for the delivery of $100,000 face value U.S. Treasury bonds whereas a corporate bond will have a par value of only $1,000. Par value will only affect the determination of the hedge ratio for the naive model. Obviously, the hedge ratio will be affected by the number of bonds one is anticipating to purchase. The market values of the respective instruments will affect the hedge ratio providing the price sensitivity model or the portfolio theory model is used to calculate the hedge ratio. These models will be discussed in Section D.

Maturity of the hedged and hedging instruments will also affect the hedge ratio, again depending on which model for calculating the hedge ratio is utilized. The conversion factor model incorporates the maturity of the cash instrument (the bond) into its hedge ratio calculation. The longer the time to maturity of the cash instrument, the smaller the hedge ratio. This model will also be discussed in Section D. The cash instrument's time to maturity will affect the hedge ratio for the price sensitivity model by affecting the duration of the cash instrument. Duration is positively related to term to maturity implying a bond with a term to maturity longer than a futures contract will have a larger hedge ratio. This is a very general rule which does not consider the coupon and yield of both instruments.

The size of the coupon for the futures contract and the cash instrument also affect the hedge ratio. The portfolio theory model considers the coupon of both instruments in calculating the hedge ratio by regressing the change in the spot price on the change in the futures price. A higher coupon implies a lower degree of price volatility for the cash instrument, holding everything else constant. Since the price sensitivity model incorporates the price, coupon, and maturity of both instruments, the yield is easily obtainable. For example, a ten percent bond selling at a discount will yield more than ten percent. If the bond would have been selling at par, the yield would have been ten percent. The formula for calculating the hedge ratio using the price sensitivity model places the yield of the cash instrument in the denominator. Clearly, the bond selling at a discount will have a lower hedge ratio than the bond selling at par provided all other factors remain the same.

It was mentioned that the risk structure of interest rates affects the hedge ratio. The price sensitivity model is the only one which considers that the riskiness of an asset matters when calculating the hedge ratio. For instance, by including the prices and yields of the hedged and hedging instruments, the price sensitivity model implicitly considers the risk of default for each instrument. Normally, a higher risk of default results in a higher yield (lower price) for a financial asset. A cross hedge involving corporate bonds and T-bond futures contracts implies a lower hedge ratio than if the cash instrument was a U.S. Treasury Bond.

The final factor to discuss which will affect the hedge ratio is the shape of the yield curve. The yield of the cash instrument and futures contract is affected by the shape and level of the yield curve. Since one of the variables used to calculate the hedge ratio for the PS model is the cash instrument's yield, obviously the hedge ratio will vary as the yield curve varies.

It should be noted that the price correlation between the hedged and hedging instruments will help to determine how reliable the hedge (a hedge ratio) will be. Correlation is normally greater when the cash instrument and the asset underlying the futures contract are the same type of asset. A future purchase of T-bonds hedged with a T-bond futures contract will probably result in a more reliable hedge than a future purchase of corporate bonds hedged with a T-bond futures contract. In the former case the price of the futures contract tends to move more closely with the price of the cash instrument than in the latter case. Hedging cash T-bonds with T-bond futures contracts is called a direct hedge whereas hedging corporate bonds with T-bond futures contracts is called a cross hedge. The correlation is usually greater for a direct hedge than it is for a cross hedge.

Determining the optimal number of T-bond futures contracts to trade is a complex process. One of the problems associated with the T-bond futures contract is that the contract is based upon a U.S. Treasury Bond with at least 15 years to maturity (or call, if callable) and an eight percent coupon. A method was devised to price the large group of deliverable issues whose characteristics (i.e., coupon rate and maturity) did not match the above requirements. A conversion factor, as stated earlier, is a number used to establish the value of each deliverable security. The Chicago Board of Trade publishes a pamphlet listing the conversion factors for T-bond futures.

The notion that U.S. Treasury Bonds with different coupons and years to maturity are deliverable against a contract raises the issue of which one should be delivered. As one would expect, the bond which is, as a practical matter, the "cheapest" relative to the other deliverable bonds will be the one most sought after for delivery. Cheapest in this case has the meaning one would normally associate with this word. The price of the futures contract will reflect the price of the cheapest to deliver cash T-bond. Not all traders will desire to hold the cheapest bond. If a trader is expecting interest rates to decline, he/she may prefer to hold a bond which has a lower coupon than the cheapest to deliver bond. Also, the issue which is the cheapest to deliver today may not be the cheapest one in the future.

To calculate which bond is the cheapest to deliver, a list is made of all deliverable bonds for the contract in question. Figure 2 lists all the bonds which qualified for delivery on June 18, 1982. This date is the last trading day for the June 1982 contract. The cheapest to deliver bond is the one which has the most positive (or least negative) basis. Basis is computed by subtracting the market price of the bond from its adjusted futures price (AFP). The AFP is equal to the price of

Coupon	Call date (Maturity)	Factor	Market price	AFP	Basis (32nds)
8.25%	5/15/00(05)	1.0233	$61.67	$60.69	(31)
11.75	2/15 01	1.3589	83.60	80.60	(96)
13.125	5/15/01	1.4931	92.03	88.56	(111)
13.375	8/15/01	1.5205	93.47	90.18	(105)
15.75	11/15/01	1.7544	107.22	104.06	(101)
14.25	2/15/02	1.6120	98.84	95.61	(103)
7.625	2/15/02(07)	0.9633	58.61	57.14	(47)
7.875	11/15/02(07)	0.9874	60.29	58.57	(55)
8.375	8/15/03(08)	1.0378	61.32	61.55	7
8.75	11/14 03(08)	1.0758	63.64	63.81	6
9.125	5/15/04(09)	1.1149	66.10	66.13	1
10.375	11/14/04(09)	1.2448	74.29	73.83	(15)
11.75	2/15/05(10)	1.3885	83.29	82.36	(30)
10.00	5/15/05(10)	1.2078	71.79	71.64	(5)
12.75	11/15/05(10)	1.4976	90.13	88.83	(42)
13.875	5/15/06(11)	1.6201	97.63	96.09	(49)
14.00	11/15/06(11)	1.6377	98.81	97.14	(54)

Figure 2. Calculating the cheapest-to-deliver bond[a]

[a]Trainer (30).

the June 1982 contract times (59-01) the conversion factor associated with each particular bond. By definition, the 8 3/8 coupon bond is the cheapest since its market price is the lowest relative to its adjusted futures price (AFP). Generally, the bond with the longest duration will be the cheapest to deliver.

What one immediately notices is that most of the bonds exhibit a negative basis. Earlier it was stated that the basis should be nearly equal to zero in the delivery month. Trainer (30) provides one explanation for the negative basis as the value of the insurance offered by hedging is so great relative to the possible losses. A portfolio manager or underwriter may be protecting the value of his/her inventories by hedging so the main concern is not the cheapest to deliver security but how much price protection the hedge will supply. This insurance is much more important to market participants than is the cheapness of the deliverable security. Another reason Trainer gives is that the meaningfulness of the basis is clouded when cross hedging.

Another idea discussed by Trainer is that from the March 1981 contract to the June 1982 contract, the basis (AFP-cash) on the most actively traded issue became more negative. He reasons this is because the sellers' alternatives created put options. For example, if an investor owns \$10 million of 14 per cents of 2011 and assuming a factor of 1.64, he/she shorts 164 T-bond futures contracts to nullify market risk. Shorting this many contracts requires \$16.4 million par value of bonds at delivery. Obviously, the investor needs an addition \$6.4 million par value to make delivery. The investor has two options.

He/she can purchase an additional \$6.4 million par value of bonds or swap the \$10 million of 14 per cents for \$16.4 million eight per cents because they have the same market value. Remembering that the last trading day for a T-bond contract is eight business days before the end of the delivery month, but the short has until the last day of the month to deliver, cash bond prices will fluctuate causing the investor to purchase the additional \$6.4 million par value if bond prices fall during these eight days or conduct the swap if market prices increase and deliver the eight per cents against the short futures position.

The various factors affecting the hedge ratio have already been discussed. Differences in the maturities, coupon rates, and yields of the asset underlying the futures contract and the cash instrument create problems which must be addressed. Suppose a portfolio manager is planning to purchase 30-year, 12 percent AAA corporate bonds and plans to hedge this purchase with T-bond futures contracts. Since the T-bond contract is based upon a 15-year, eight percent coupon, certain adjustments will have to be made regarding the number of contracts to trade. The model to be used in this study will be presented followed by a discussion of the factors affecting the hedge ratio.

The general formula used to calculate the hedge ratio derived by Kolb and Chaing (21) is

$$N = \frac{\overline{R}_j \ P_i \ D_i}{\overline{R}_i \ FP_j \ D_j} \qquad (3)$$

where N is the number of contracts to trade, $\overline{R}_j$ = 1 + the expected interest rate on the asset underlying futures contract j, $\overline{R}_i$ = 1 + the yield to maturity expected for asset i, P_i = the price of asset i, FP_j = the price for futures contract j at maturity, D_i = the duration of asset i, and D_j = the duration of the asset underlying futures contract j. It is important to realize P_i, D_i, FP_j, and D_j are all values expected to be realized at the termination of the hedge. These values do not change over the course of the hedge because the yield curve is assumed to remain flat. Assume the objective of hedging is to leave the hedger's initial position unchanged. Duration will be discussed in Chapter III. At this point it is sufficient to define duration as a weighted average time to maturity measured in years.

A brief examination of equation (3) reveals how the coupon rate, maturity and yield of the cash and futures instruments will affect the hedge ratio. A higher yield on the cash instrument relative to the futures contract, ceteris paribus, will cause the value of N to decline. Assume now that the yields are equal but the maturities are different, as could be caused by cross hedging. If the cash instrument has the longer maturity, the hedge ratio will be greater than if the asset underlying the futures contract had the longer maturity. Coupon rates are inversely related to duration so their affect upon N is just the opposite as the affect of maturity.

Consideration must also be given to the riskiness of each instrument involved. AAA corporate bonds will have a different hedge ratio than will BBB or U.S. Treasury Bonds. These will affect the hedge ratio by affecting the yield on the cash instrument and the futures contract. The affect that the yield on the cash instrument and futures contract has on the hedge ratio has already been discussed.

D. Methods for Calculating the Hedge Ratio

Four methods for calculating the hedge ratio will be presented. The first method, called the price sensitivity model (PS), takes into account all but one of the factors previously mentioned. This method can be compared to the three other methods. It can be shown that the PS model is the one that consistently provides the investor with the closest approximation to a perfect hedge.

The basic situation involves a future purchase of $10 million of AAA corporate bonds. A hedging period of three months is assumed. Fearing a decrease in interest rates between now (March 1) and the time the actual purchase takes place (June 1), the portfolio manager can hedge this investment by purchasing T-bond futures contracts on March 1 and selling them on June 1. Ideally, his/her expectations will be realized so that the gain from purchasing the futures contracts will exactly offset the opportunity loss associated with the decline in interest rates.

All four methods will be compared in four different cases. Cases 1 and 2 involve the purchase of a 20-year, six percent bond portfolio whereas cases 3 and 4 examine the situation when the bonds have a ten percent coupon. Also, in cases 1 and 3 expectations are realized (interest rates decline) but in cases 2 and 4 interest rates increase.

Table 1 shows the yields and prices for the cash and futures instruments. It also shows the associated gain (loss) due to the change in the yield. The yield and price data for the corporate bonds were obtained from the Thorndike Encyclopedia of Banking and Financial Tables. T-bond futures bond data were taken from the Wall Street Journal. The Wall Street Journal assumes the asset underlying the futures contract is an eight percent, 20-year U.S. Treasury Bond.

1. Price sensitivity (PS) model

The model which is able to account for all but one of the factors previously described is the PS model. Developed by Kolb and Chaing (21), this model has one flaw in that there is an implicit assumption of a flat term structure. As will be shown later, the empirical evidence indicates this assumption does not hinder its effectiveness in practice.

The goal of the PS strategy is to avoid a change in the value of the portfolio. Mathematically, this may be written as

$$P_i + P_j\,(N) = 0 \tag{4}$$

where P_i and P_j represent the values of the asset to be hedged and the futures contract respectively, and N represents the number of futures contracts to trade. Due to the fact that the risk, maturity, and coupon structure of the cash instrument and the futures contract will not be equal, the problem lies in determining the value for N so the price sensitivity of asset i, given a change in interest rates, is equal to the price sensitivity of futures contract j times N.

Table 1. AAA corporate bond and T-bond futures prices under various yield scenarios

			T-bond futures	
Case 1: 6%, 20-year corporate				
Date	Yield	Price	Yield	Price
March 1	12.201	$539.35	11.201	$74,656.25
June 1	11.789	558.64	10.789	77,312.50
		(19.29)		2,656.25
Case 2: Interest rates increase for case 1 bond				
Date	Yield	Price	Yield	Price
March 1	12.201	$539.35	11.201	$74,656.25
June 1	12.500	526.00	11.500	72,812.50
		13.35		(1,843.75)
Case 3: 10%, 20-year corporate				
Date	Yield	Price	Yield	Price
March 1	12.201	$836.56	11.201	$74,656.25
June 1	11.789	863.59	10.789	77,312.50
		(27.03)		2,656.25
Case 4: Interest rates decrease for case 2 bond				
Date	Yield	Price	Yield	Price
March 1	12.201	$836.56	11.201	$74,656.25
June 1	12.500	817.70	11.500	72,812.50
		18.86		(1,843.75)

Duration is an integral part of this model and has already been defined in equation (1). Another way to define it is the negative price elasticity of a bond with respect to a change in the discount factor (i).

$$D = -\frac{dP/P}{di/(1+i)}. \tag{5}$$

To calculate N, the number of contracts to trade, we solve

$$\frac{dP_i}{dR_f} + \frac{dP_j}{dR_f} N = 0 \tag{6}$$

where dR_f equals 1 + the risk-free rate.

A closer look at the general solution for calculating N when the hedging instrument is a T-bond futures contract is in order. Recall earlier that when the coupons and maturities are different for the cash instrument and the asset underlying the futures contract, the hedge ratio will be affected. Suppose a portfolio manager plans to purchase 20-year AAA corporate bonds in the near future. These bonds have coupons of six percent and ten percent. The purchase will be hedged with T-bond futures contracts. Since the cash and futures differ as to their coupons and maturities, the correct number of contracts to trade is given by equation (3).

The values of N calculated by using equation (4) are listed in Table 2. The price and yield data were taken from Table 1. Duration values were derived using the data in Table 1 assuming semiannual compounding. To calculate the number of contracts to trade, simply

Table 2. Calculation of the hedge ratio using the PS model[a]

$$PS_{1,2} = \frac{1.11201\ (\$539.35)\ 8.951}{1.12201\ (\$74{,}656.25)\ 8.865} = 0.00723834$$

$$PS_{3,4} = \frac{1.11201\ (\$836.56)\ 8.127}{1.12201\ (\$74{,}656.25)\ 8.865} = 0.01018109$$

[a] $PS_{1,2}$ applies to cases 1 and 2, the same for $PS_{3,4}$.

multiply the value from equation (3) times the number of bonds to be purchased. For example, in case 1 this equals 0.00723834 times 18,541 ($10 million/ $539.35) or 134 contracts. The purchase of 134 T-bond contracts yields a gain of $355,938. This is found by multiplying the gain on each contract ($2,656.25) times 134. The opportunity loss on the investment is equal to the increase in the price of the bond ($19.29) times the number of bonds purchased (18,541) or $357,656. By subtracting the loss from the gain, the net result is a loss of $1,718. This is roughly 0.02 percent of the future investment. Table 3 lists the results for all methods and cases.

It has been shown that the PS strategy does provide a close approximation to a perfect hedge. The three other methods will be compared to this method. A problem develops in making a comparison because the objective of each model is not the same. For example, the PS model

Table 3. Comparison of the hedging models

	PS	Naive	Conversion factor	Portfolio theory
Case 1:				
1. Contracts purchased	134	100	113	167
2. Futures gain	$355,938	$265,625	$300,156	$443,594
3. Bond loss (opportunity)	(357,656)	(357,656)	(357,656)	(357,656)
4. Net effect	($1,718)	($92,031)	($57,500)	$85,938
Case 2:				
1. Contracts purchased	134	100	113	167
2. Futures loss	(247,063)	(184,375)	(208,344)	(307,906)
3. Bond gain (unrealized)	247,522	247,522	247,522	247,522
4. Net effect	$459	$63,147	$39,178	($60,384)
Case 3:				
1. Contracts purchased	122	100	83	108
2. Futures gain	324,063	265,625	220,469	286,875
3. Bond loss (opportunity)	(323,117)	(323,117)	(323,117)	(323,117)
4. Net effect	$946	($57,492)	($102,648)	($36,242)
Case 4:				
1. Contracts purchased	122	100	83	108
2. Futures loss	(224,938)	(184,375)	(153,031)	(199,125)
3. Bond gain (unrealized)	225,452	225,452	225,452	225,452
4. Net effect	$514	$41,077	$72,421	$26,327

strives to leave the hedger's initial position unchanged whereas the portfolio theory model tries to minimize the risk of price changes in the portfolio.

2. The naive model

Simplicity is the most attractive feature of the naive model. It may also be called the equal and opposite position model because the strategy involves equating the size of the futures position to the size of the future cash position. For a planned investment of $10 million, the model calls for the purchase (case 1) of 100 T-bond contracts. The results are presented in Table 3. Notice that this model produces very inconsistent results.

One problem with using this model is that the number of contracts traded is a constant. If interest rates decrease 10 or 200 basis points, the futures position does not change. The method only considers the face value of the hedging instrument. It seems that this model should only be used as a starting point for further research and not as a strategy for actual hedging.

3. The conversion factor model

This method for calculating the optimal number of futures contracts to trade uses the conversion factor as its key element. If the cash instrument was an eight percent, 15-year bond, the conversion factor would be 1.00. This is rarely the case, however, which is why the factor model was developed. This model provides a rule for hedging a bond purchase or sale when the coupon rate is not eight percent and the maturity (call) is not 15 years.

The first step is to calculate the number of contracts using the naive model. As noted in the previous section, this would be 100 T-bond contracts. Now, to adjust the number of contracts for the characteristics (coupon, maturity) of the specific bonds to be purchased, divide the number of contracts to be purchased under the naive model by the conversion factor. For case 1 this means to divide 100 by 0.885. Again, Table 3 presents the results for all examples.

4. The portfolio theory model

Portfolio theory evaluates the return from an investment given its degree of risk. Ederington (12) was the first person to integrate portfolio theory and hedging in the financial futures market. He shows for a given spot position Xs, (par value) the proportion which should be hedged is given by

$$b = -Xf/Xs \tag{7}$$

where Xf is the the par value of the futures position. The negative sign accounts for the fact that the futures position (long) and the spot position (short) are not the same so that b will usually have a positive value.

The hedger wishes to minimize the risk associated with a particular investment. This minimized value of b, b* is

$$b^* = \theta sf/\theta^2 f \tag{8}$$

where $\theta^2 f$ is the subjective variance of the futures price change during the hedging period, and θsf is the covariance between the price change of the spot and futures instruments. The estimate of b* provides the hedger with the number of futures contracts he should buy (case 1) to minimize the price risk caused by fluctuating interest rates.

Estimating b* is accomplished by regressing time series data of price changes in the spot position (ΔPs) on price changes in the futures position (ΔPf).

$$(\Delta Ps) = a + b^*(\Delta Pf). \tag{9}$$

The slope coefficient becomes the estimate for b*.

Hill and Schneeweis (18) regressed price changes of Moody's AAA corporate bond index on price changes of the T-bond futures contract. The data set consisted of month-end to month-end differences in contract values fron August 1977 to December 1979. Their average estimate of b* was 0.90. Since the example in Table 3 assumes the purchase of 18,541 bonds, Xs will equal $18,541,000. This generates an Xf of $16,686,900 (0.90 times $18,541,000). The number of T-bond contracts to purchase using this method will be 167 for the first case. Table 3 shows the results for all four cases.

This method contains a few problems which may not be easy to deal with effectively. First of all, the regression technique can only be performed if ample data are available. Secondly, b* gives the best hedge ratio ex post but the ratio may not be appropriate for future hedging activity. The time period used to estimate b* will greatly affect its value and there is no guarantee that a longer time period will more acurately estimate b* than a shorter one.

Table 3 shows that it makes no difference whether interest rates increase or decrease, the PS model consistently provides the investor (hedger) with the closest approximation to a perfect hedge for the time period simulated. The size of the net effects for this model are trivial relative to the size of the investment. There was more variation in the net effect of the PS method when the bond to be purchased had a six percent coupon as opposed to the ten percent bond. This is due to the fact that for two bonds with the same maturity, the one with the lowest coupon will have the more volatile price. Based on these comparisons, the PS model appears to be the most efficient.

III. DURATION AND IMMUNIZATION THEORY

A. Properties

Duration provides a more accurate description for the price volatility of a coupon bond than does the term to maturity for a given change in the discount rate. One reason is that the term to maturity only considers the timing of the final payment a bond holder receives. The timing of the semiannual coupon payments is ignored. A larger coupon normally implies a lower degree of price volatility than a smaller coupon assuming equal yields and terms to maturity and assuming an equal change in the discount factor for both bonds. Term to maturity is not affected by the size of a bond's coupon but duration is inversely affected. This is why duration, as oppossed to term to maturity, is a more complete description of a bond's price volatility. Also, recall from equation (1) that duration is a function of the yield to maturity and time to maturity of a bond. The former affects duration inversely and the latter directly.

Any type of coupon bond has a duration less than or equal to its maturity while the duration of a zero coupon bond is equal to its maturity. This is shown in Figure 3. Notice the behavior of duration is different for discount versus par or premium bonds. The duration of a par or premium bond (coupon > yield to maturity) increases monotonically for an increase in its maturity and reaches a maximum at the inverse of yield to maturity.[1] The behavior of duration for a discount bond is

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[1]The duration for any coupon bond is bounded at perpetuity by $i+p/i \cdot p$, where i is the yield to maturity and p is the number of compounding periods per year.

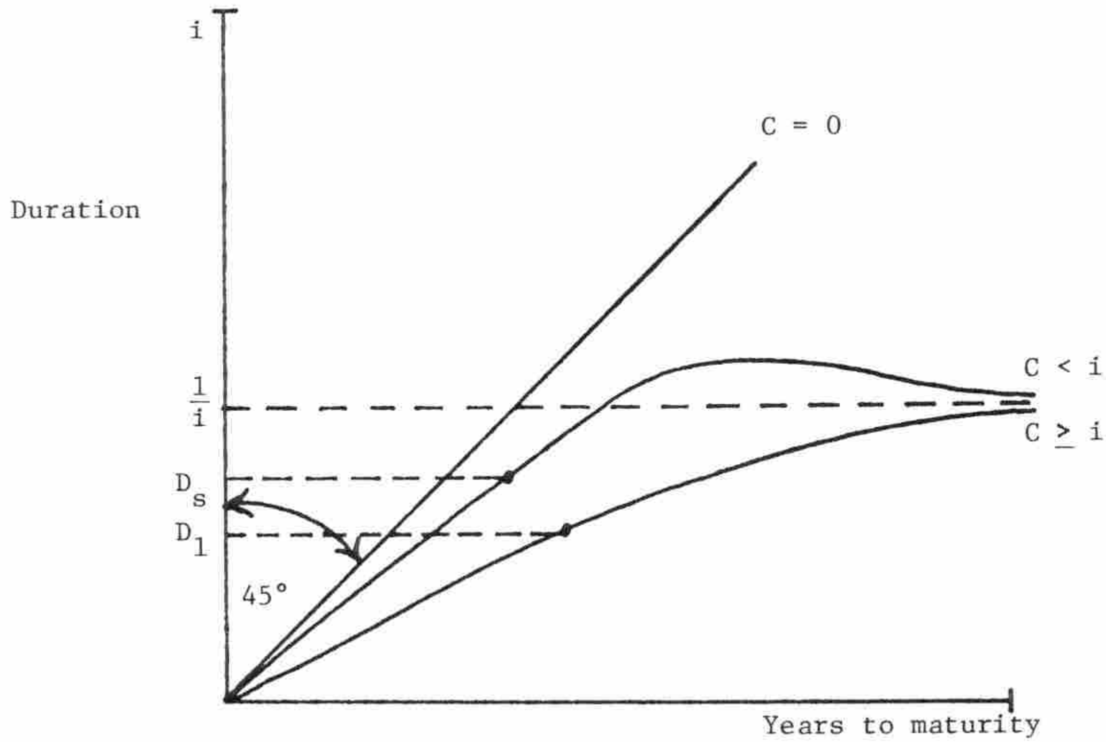

Figure 3. Graphical representation of duration behavior

slightly more complex. Its duration increases more rapidly than does the duration for a par or premium bond until its maximum is reached. The maximum duration is obtained when:

$$n = 1/i + 1/i-c + i/i-c + i-c/ic(1 + i) \tag{10}$$

where c = the coupon rate, i = the yield to maturity and n = the term to maturity. Figure 3 helps to explain how the price of a shorter-term low coupon bond can be more volatile than the price of a longer-term high coupon bond during periods of high interest rates. It is easy to see from Figure 3 that the duration of the low coupon short-term bond, Ds, may be larger than the duration of the longer-term higher coupon bond, Dl. These differences in the behavior of duration have helped to explain some anomalies in bond pricing which have confused some people in the past.

By rewriting equation (5) in terms of first differences, one obtains

$$\Delta P/P = -D\Delta i/(1 + i), \tag{11a}$$

$$\Delta P = -PD\Delta i/(1 + i) \tag{11b}$$

where P = the price of the bond, D = duration and i = yield to maturity. This is actually a theorem proven by Hopewell and Kaufman (20). Their theorem is restated here: "For a given basis point change in market yield, percentage changes in bond prices vary proportionately with duration and are greater, the greater the duration of the bond." This theorem was developed in response to one of Malkiel's theorems which stated that changes in bond prices are greater, the longer is the bond's term to maturity. Hopewell and Kaufman pointed out that since duration does not always increase as the maturity of a bond increases, changes in bond prices will not always increase either.[1] Table 4 shows values of duration for different coupons, yields, and maturities.

Knowing how to calculate duration is useful for evaluating investment opportunities. Here is a simple example. Let us assume the yield on a high grade new issue ten-year bond is 9.0 percent and the yield on another high grade new issue 20-year bond is 11.5 percent. The duration of the ten-year bond is 7.12 whereas the duration of the 20-year bond is 9.65. By calculating their ratio, 0.734, the investor finds that the price volatility of the shorter-term issue, with respect to a change in the discount rate, is 73.4 percent of the volatility of the longer-term issue. Since the absolute difference in yields is large when compared to the differences in durations, an investment in the 20-year bond may be advisable.

McEnally (26) showed that another use for measuring duration is to get an approximation of the interest rate sensitivity of a fixed income security. He modified equation (11a) and obtained

$$(\Delta P/P)\ 100\% = \Delta i \cdot 100\%\ (-D/(1 + i)) \tag{11c}$$

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[1]This only occurs for long maturity (in excess of 50 years), high yield bonds.

Table 4. Duration values of bonds with semiannual coupons[a]

Years to maturity	Promised yield to maturity[b] 4 percent			6 percent			8 percent		
	Coupon rate 4%	6%	8%	Coupon rate 4%	6%	8%	Coupon rate 4%	6%	8%
1	0.990	0.986	0.981	0.990	0.985	0.981	0.990	0.985	0.981
5	4.581	4.423	4.290	4.558	4.393	4.254	5.533	4.361	4.218
10	8.339	7.859	7.497	8.169	7.662	7.286	7.986	7.454	7.067
20	13.951	12.876	12.181	12.980	11.904	11.232	11.966	10.922	10.292
50	21.980	20.629	19.903	17.129	16.273	15.829	13.466	12.987	12.743
100	25.014	24.535	24.293	17.232	17.120	17.064	13.029	13.008	12.995
∞	25.500	25.500	25.500	17.167	17.167	17.167	13.000	13.000	13.000

[a]Fisher and Weil (13).

[b]Yields and rates recorded as percentage per annum, semiannual compounding.

where $-D/(1 + i)$ is equal to the adjusted duration. Suppose the duration of a ten percent, five-year bond priced at par is 4.17 years. The adjusted duration is equal to -3.79 (-4.17/1.10). Now assume the discount rate increases by 50 basis points. The price of the bond should decline by 1.895 percent (3.79 0.005). The actual price decline is 1.91 percent ($980.90). Accuracy improves for smaller changes in the discount factor.

Most bonds issued today have a call provision attached to them. The maturity of the bond will be reduced substantially if the bond is called. This implies the duration of a callable bond may be less than previously anticipated. Another feature which will decrease the duration of a bond is a sinking fund. This is because the cash flows in the later years will be increased giving less weight to the final payment. The difference between the two is that a corporation is legally bound by the sinking fund. Also, not all investors will be affected by a sinking fund. An investor should be aware of these two features and their effect upon the price volatility of the portfolio.

B. Calculating Duration

Using equation (1) we can compute the duration of any bond rather easily. Table 5 shows how to calculate the duration for a 5 year, 10 percent bond priced at par. The PVIF is the present value interest factor for a bond yielding ten percent. Multiplying column 1 (year) by column 5 (PV of price) results in the portion of duration attributable to that year's cash flow. Recall from Figure 3 that as years to maturity

Table 5. Duration of a ten percent coupon five-year bond priced at par

Year	Cash flow	PVIF	PV of flow	PV of price	Duration
1	\$100	.9091	\$90.91	.09091	.09091
2	100	.8264	82.64	.08264	.16528
3	100	.7513	75.13	.07513	.22539
4	100	.6833	68.33	.06833	.27330
5	1100	.6209	682.99	.68299	3.41495
Sum			\$1000.00	1.00000	4.16983
Duration					4.17 years

increase, the duration of a bond sold at par increases at a decreasing rate. The PVIF becomes very low for long maturity bonds. The cash flows for the final years contribute very little to the total duration of the bond. This can be seen by comparing the duration values in Table 4 for 50 and 100 years to maturity.

A shorthand approximation for calculating duration has been provided by Macaulay (25).

$$D = \frac{R}{R-1} - \frac{QR - T(1 + Q - QR)}{R^T - 1 - Q + QR} \qquad (12)$$

where R = 1 + yield to maturity, Q = face value/semiannual coupon payment, and T = years to maturity times the number of coupon payments per year. The result will be in half-years so it must be divided by two. For example, a five-year, ten percent coupon bond with a face value of $1,000 yielding ten percent will have a duration of

$$D = \frac{1.05}{.05} - \frac{20(1.05) - 10[1 + 20 - 20(1.05)]}{1.05^{10} - 1 - 20 + 20(1.05)},$$

$$= 8.110 \text{ years (semiannual)},$$

$$= 4.055 \text{ years.}$$

Equation (12) may be easily stored in a computer so that by entering the bond's yield, maturity, coupon, and face value, one could obtain its duration.

C. Alternative Measures of Duration

Although Macaulay was the first person to define duration, others have developed more complex ways to measure it. Equation (1), as pointed out earlier, assumes a flat yield curve and equal interest rate changes for all maturities (i.e., flatness is maintained). This measure of duration is commonly referred to as D1 in the literature. This study will discuss two other duration measures. These are slightly more complex than D1. There are more measures of duration, some very complex, but these will not be discussed here. The reason these more complex measures will not be discused is that it has been shown they do not outperform the simpler measures in practice and are not widely used in the literature.

A second measure of duration, developed by Fisher and Weil (13), differs from Macaulay's in that they do not assume a flat yield curve. Mathematically,

$$D2 = \frac{\sum_{n=1}^{m} \frac{Cn}{\prod_{t=1}^{n} (1+r_t)^n} + \frac{Am}{\prod_{t=1}^{m} (1+r_t)^m}}{\sum_{n=1}^{m} \frac{C}{\prod_{t=1}^{n} (1+r_t)^n} + \frac{A}{\prod_{t=1}^{m} (1+r_t)^m}} \qquad (13)$$

where C is the coupon, n is the number of periods to the coupon payment, A is the face value of the bond, m is the number of periods to maturity, r_t is the one period forward rate in time t, and π is the multiplication operator. This measure assumes an additive random shock to the term structure as did D1. D2 and D1 are similar because they both assume the yield curve maintains its shape after the random shock occurs though D2 does not assume a flat yield curve.

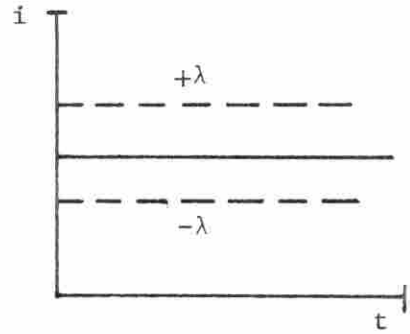
The final measure presented is the most complex. For D3 the term structure does not shift in a parallel fashion. The slope is changed but the general direction is preserved so that instead of an additive interest rate shock as assumed in D1 and D2, D3 assumes the shock is multiplicative. If the initial term structure was described by $[1 + h(0,t)]$, the new term structure will be described by $(1 + \lambda)[1 + h(0,t)]$ where $\lambda > 0$ is a random variable. The equation for D3 is

$$D3 = \frac{\sum_{t=1}^{m} tC[1 + h(0, t)]^{-t} + mA[1 + h(0, m)]^{-m}}{\sum_{t=1}^{m} C[1 + h(0, t)]^{-t} + A[1 + h(0, m)]^{-m}} \qquad (14)$$

where $h(0,t)$ = the zero coupon yield equivalent for the period spanning 0 to t. Figure 4 shows how the types of random interest rate shocks which will affect the term structure for D1, D2, and D3. Recall that D1 assumes the term structure of interest rates is flat initially and remains flat after the shock. D2 is assumed to have shape but the interest rate shock does not change the shape of the yield curve, only its level (i.e., parallel shift). The basic shape of the yield curve is preserved in D3 but the shift is not parallel. If the yield curve was

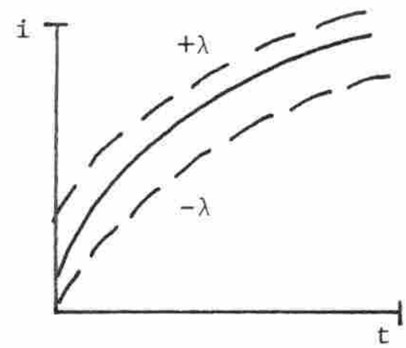
D1: Macaulay additive

$$\frac{\Delta\lambda}{\Delta t} = 0$$



D2: Fisher-Weil additive

$$\frac{\Delta\lambda}{\Delta t} = 0$$



D3: Bierwag multiplicative

upward-sloping term structure $\quad \frac{\Delta\lambda}{\Delta t} > 0$

downward-sloping term structure $\quad \frac{\Delta\lambda}{\Delta t} < 0$

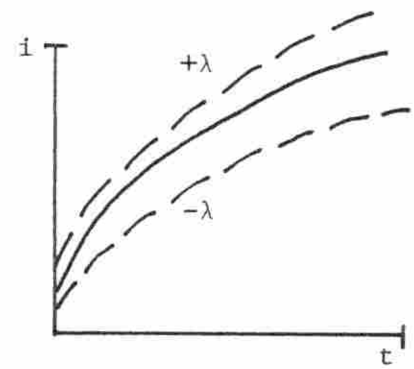


Figure 4. Graphical representation of the interest rate shocks for various stochastic processes

upward-sloping originally, it will remain upward-sloping after the interest rate shock but the new yield curve will not be parallel to the original one.

Values of D1, D2, and D3 for an upward-sloping and a downward-sloping term structure are presented in Table 6. The bonds have five percent and ten percent coupons and maturities ranging from one to 25 years. Notice all three measures of duration produce approximately the same values regardless of the size of the bond's coupon. Bierwag and Kaufman (5) have stated D1 may be a first approximation for D2 and D3. An advantage of D1 is that a forecast of the one period forward rates is not needed since D1 is a function of the yield to maturity for the bond. D1 also has an advantage in that an assumption about the nature of the random shock affecting interest rates is not needed.

D. Immunization and Stochastic Process Risk

Central to the success of an immunization strategy is the ability of the investor to correctly identify the stochastic process generating unexpected changes in interest rates. A stochastic process may be defined as random changes in individual yields which follow a general predetermined pattern. An example of one such pattern is that unexpected changes in interest rates will affect shorter-term maturities more than longer-term ones. Since each method for computing duration is based upon a specific stochastic process, incorrectly matching a measure of duration and the underlying stochastic process will probably result in a realized

Table 6. Duration values for alternative measures[a]

Maturity (years)	Yield[b]	5 percent coupon D1	5 percent coupon D2	5 percent coupon D3	10 percent coupon D1	10 percent coupon D2	10 percent coupon D3
		Upward-sloping yield curve					
1	6.10	0.99	0.99	0.99	0.98	0.98	0.98
2	6.20	1.93	1.93	1.93	1.87	1.87	1.87
3	6.30	2.82	2.82	2.82	2.68	2.68	2.68
4	6.40	3.67	3.66	3.67	3.43	3.43	3.45
5	6.50	4.47	4.46	4.48	4.13	4.12	4.16
6	6.60	5.22	5.22	5.25	5.37	5.35	5.42
7	6.70	5.94	5.92	5.99	5.93	5.90	6.00
8	6.80	6.61	6.59	6.66	6.45	6.40	6.53
9	6.90	7.24	7.20	7.31	7.39	7.29	7.49
10	7.00	7.83	7.78	7.92	6.93	6.86	7.03
15	7.50	10.23	10.00	10.36	8.92	8.68	9.04
20	8.00	11.82	11.24	11.87	10.36	9.80	10.37
25	8.50	12.81	11.74	12.61	11.41	10.42	11.18
		Downward-sloping yield curve					
1	8.50	0.99	0.99	0.99	0.98	0.98	0.98
2	8.40	1.93	1.93	1.93	1.86	1.86	1.87
3	8.30	2.81	2.81	2.81	2.67	2.67	2.67
4	8.20	3.65	3.65	3.65	3.41	3.41	3.41
5	8.10	4.44	4.44	4.43	4.09	4.10	4.06
6	8.00	5.18	5.19	5.16	4.71	4.72	4.68
7	7.90	5.88	5.89	5.84	5.29	5.31	5.24
8	7.80	6.53	6.55	6.47	5.82	5.85	5.76
9	7.70	7.14	7.17	7.06	6.31	6.36	6.23
10	7.60	7.71	7.76	7.61	6.77	6.84	6.67
15	7.10	10.02	10.24	9.81	8.64	8.89	8.48
20	6.60	11.65	12.22	11.37	10.01	10.59	9.84
25	6.10	12.79	13.91	12.51	11.04	12.13	10.94

[a]Bierwag and Kaufman (5).

[b]Implied zero coupon yields to maturity.

return which is less than the promised return. In other words, the investor is exposed to stochastic process risk.

Immunization was first coined by the British actuarist F. M. Redington (28). He found the value of an insurance firm is immune to unexpected changes in interest rates when "the mean term of the value of the assets proceeds...[equals] the mean term of the value of the liability-outgo." The mean term is defined to be the first derivative with respect to interest rates. Later, Fisher and Weil (13) found a portfolio could be immunized if its duration is set equal to the length of the investor's holding period. They stated that a portfolio of bonds is immunized for the predetermined holding period if its value at the end, regardless of the change in interest rates, is no less than its value would have been given constant interest rates. More recently, Leibowitz and Weinberger (24) have developed a procedure for active bond management. Active management involves setting the duration of the portfolio either greater than, or less than, the length of the holding period. This procedure, called contigent immunization, allows the bond manager to earn a minimum return on the portfolio even when interest rates move against his/her position. The portfolio is in an active mode until it becomes necessary to switch to an immunization mode.

1. Classical immunization

In theory, a classical immunization strategy involves setting the duration of the portfolio equal to the length of the investor's holding period. This assures the investor that for an unexpected change in interest rates, the gain (loss) from the reinvestment of the coupons will equal the loss (gain) in the value of the portfolio. The extreme case will involve the purchase of a zero coupon bond whose maturity equals the length of the investor's holding period. In this case, the length of the holding period and the time to maturity (duration) of the bond decrease at exactly the same rate. Perfect immunization is achieved. This has an added advantage in that no reshuffling of the portfolio is necesary. The problem with this method is the supply of zero coupon bonds is very thin, consequently the complexity of the immunization strategy is increased.

Bierwag, Kaufman, and Toevs (7) have found the three measures of duration explained in section C all perform equally well when the assumed correct measure is D2. They tested D1, D2, and D3 assuming all bonds have a five percent coupon, a planning period of five years and a promised return of 5.91 percent. Three barbell portfolios (5-40, 1-10, and 6-7) were immunized and then interest rate shocks varying from +360 to -360 basis points occurred.[1] These are then compared to a portfolio consisting of bonds whose maturity equals the holding period. The simulations assume an upward-sloping term structure initially described by

Term (years)	1	3	5	10	15	20
Interest rate (%)	4.3	5.5	5.9	6.7	6.8	6.8

The results in Table 7 show realized returns less promised returns in basis points per annum.

[1]A barbell portfolio consists of securities having long and short maturities with no securities having an intermediate maturity. For example, a 5-40 barbell consists of bonds maturing in five and 40 years, respectively.

Table 7. Realized return less promised return[a,b]

Interest rate shock[c] (basis points)	D1	D2	D3	Maturity matched
		5-40 barbell		
-360	25.3	22.6	25.1	-36.8
-180	5.3	5.2	4.2	-18.8
-90	1.2	1.2	0.7	-9.5
-20	0.1	0.0	0.0	-2.1
+20	0.1	0.1	0.1	2.1
+90	1.0	1.4	1.1	9.7
+180	3.9	4.6	4.0	19.5
+360	14.5	15.7	14.6	39.8
		1-10 barbell		
-360	21.9	18.6	21.0	-36.8
-180	5.4	4.4	5.0	-18.8
-90	1.3	0.5	1.2	-9.5
-20	0.1	-0.1	0.0	-2.1
+20	0.1	0.3	0.1	2.1
+90	1.3	2.1	1.6	9.7
+180	5.4	6.9	5.7	19.5
+360	20.8	24.1	21.6	39.8
		6-7 barbell		
-360	2.2	1.4	2.1	-36.8
-180	0.6	0.2	0.5	-18.8
-90	0.1	0.0	0.2	-9.5
-20	0.0	0.0	0.0	-2.1
+20	0.0	0.0	0.0	2.1
+90	0.1	0.0	0.2	9.7
+180	0.6	1.0	0.6	19.5
+360	2.4	3.1	2.4	39.8

[a]Bierwag, Kaufman, and Toevs (7).

[b]Figures recorded in basis points per annum.

[c]Assume actual interest rate shock is described by D2.

Before the popularity of immunization strategies, maturity matching was the most frequently used method. The maturity matched column is presented to provide a means of evaluating whether or not the immunization strategies are more effective in providing the investor with realizing a rate of return equal to the promised return. Notice that in only one instance the promised return is less than the realized return. Also, note that the difference between the realized and promised return increases as the absolute size of the interest rate shock increases. One important finding is the closer the individual durations (6-7 barbell) in the portfolio are to the length of the holding period, the closer the realized return is to the promised return. This means that the probability of perfect immunization is increased by concentrating the individual durations in the portfolio as close as possible to the holding period. Another advantage of this strategy is if the underlying stochastic process is incorrectly identified, the possibility of earning a return less than the promised return is reduced. In summary, immunizing a portfolio is a function of the specific measure of duration used, the length of the holding period, and the composition of the portfolio.

There are two reasons why duration condensing is a successful immunizing technique. When the underlying stochastic process is not correctly identified, the investor becomes exposed to stochastic process risk. This risk can be reduced by adjusting the cash flows of the portfolio so they closely resemble a zero coupon bond whose maturity equals the investor's holding period. This zero coupon bond, as mentioned earlier, will immunize the investor regardless of the type of stochastic process which actually prevails. The portfolio whose individual durations are condensed around the length of the holding period will generate the closest approximation to desired cash flow pattern. This has already been shown in Table 7. The second reason is if the stochastic process assumed by the investor does not fullfill equilibrium conditions, the more spread out the individual durations, the more likely the realized return is to be further from the promised return.

2. Active versus passive management

There are two possible immunization strategies a portfolio manager may want to use, depending upon the needs of his/her clients--a passive strategy which has a goal of earning the current "risk-free" rate or an active strategy. The active strategy chosen depends upon the manager's belief as to how interest rates will vary. Babcock (1) estimated the relationship betwen the expected rate of return on a bond and future interest rates using this equation.

$$E(R) = i + (1-D/PL)\Delta i \tag{15}$$

where $E(R)$ = expected rate of return on bond j, D = Macaulay's measure of duration (D1), PL = the investor's planning horizon, i = promised yield to maturity for PL, and Δi = difference between the predicted and promised yield to maturity for PL.

Suppose very risk-averse investors feel content earning the current yield to maturity for their portfolio. They can assume a theoretically risk-free position by setting the duration of their portfolio equal to the length of the planning period. The second term in equation (15) drops out so these investors will expect to earn i over their planning horizon.

Now assume that another group of investors are not as risk-averse and do not mind trying to increase their return (over i) at the expense of a possible loss. Depending upon beliefs as to the direction of the change in interest rates, they will set the duration of the portfolio either greater than, or less than, the length of the planning period. If these investors feel interest rates are going to rise, the correct action will be to decrease the sensitivity of the portfolio to changing interest rates (i.e., $D < PL$). Equation (15) shows that this will increase the expected return on the investment. The opposite would be true if interest rates are expected to fall. If that prediction is incorrect, the realized return will be less than i. The maturity matched column of Table 7 shows that if the duration of a portfolio is less than the holding period, a negative interest rate shock decreases its realized return. Similarily, a positive shock increases the realized return above the promised return. The difference between D and PL is a function of the investor's degree of risk aversion and the strength of his/her belief as to the future course of interest rates. Another alternative would be to assume a risky position, say $D > PL$, but at the same time hedge part of this position by purchasing T-bond futures contracts.[1] If predictions are correct the investor will increase the return on the portfolio but at the same time will have lost money on the futures contracts. The size of D relative to PL depends upon the intensity of beliefs concerning future movements of interest rates.

3. Problems

Throughout this chapter it has been stated that by immunizing a portfolio of bonds, an investor is able to insulate the value of his portfolio from changing interest rates. Several criticisms of this duration-immunization (DI) strategy have been raised. The criticisms and problems relate to the theoretical, as well as the pragmatic, aspects.

On a theoretical note, the three measures of duration discussed earlier have returns which increase as the size of the random shock, λ, increases. This property creates the opportunity for realizing a greater return than originally promised by the term structure simply because interest rates change during the holding period. Regardless if rates increase or decrease, the realized return will increase as long as the duration of the portfolio is equal to the investor's holding period.

A second theoretical problem involves the violation of a general equilibrium condition using the measures of duration previously mentioned. In competitive markets, there exists a direct relationship between the degree of risk and the return for a particular investment.

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[1]An alternative strategy would have involved the purchase of T-bond call options.

Riskless arbitrage opportunities should not exist. One may, however, earn a higher return (more current income), without assuming any more risk, by swapping low coupon bonds for higher coupon ones while leaving the duration of the portfolio unchanged. The complex duration measures mentioned earlier do not violate this equilibrium condition. These measures, however, are not widely used in practice or in the literature.

Macaulay's duration, which assumes a flat yield curve, seems to out perform all other measures. It seems inconsistent that a flat yield curve model out performs other measures which have been developed to account for a yield curve with "shape," when all indications are the yield curve does have "shape."

Another problem is that the measures explained in section C are called single-factor duration models. This means the stochastic process generating changes in interest rates can be used to derive a measure of duration. The single-factor statistical characteristic used to describe the direction and amount amount of the term structure shift is called duration. For any single-factor duration to be accurate, changes in all interest rates must be perfectly correlated. In other words, the shape of the yield curve will be preserved. If this is not the case, then more than one measure of duration would be needed to describe the underlying stochastic process. This would greatly increase the complexity of the problem.

Equation (11a) holds only for a very small change in interest rates or bond prices. Recently, the volatility of interest rates has increased dramatically. The effect of this increased volatility has been to increase the size of any error which may result from using (11a) as a measure of bond price volatility. Yawitz and Marshall (32) point out this equation holds only if di is the same for all bonds. Since there does not exist a general relationship between yield volatility and duration, only in rare circumstances will two bonds with equal durations be equally risky.

Note that equation (1) is a function of the yield to maturity of the bond. Thus, only for a change in the yield to maturity of a bond, does equation (1) provide an accurate measure for the change in its price. This results from the fact that for a flat term structure, the yield to maturity is equal to the average of the one period discount rates. A change in the yield to maturity may result from a wide variety of changes in the discount rates when the term structure is not flat. Macaulay's measure of duration does not describe how a bond's price will change when discount rates change, only when its yield to maturity changes.

As time passes, the duration and the length of the holding period will decrease. A problem develops because only in the extreme case (a zero coupon bond whose maturity is equal to the holding period) will both of these decrease at the same rate. Periodically, the portfolio will need to be adjusted so its duration remains equal to the time remaining in the holding period. Adjusting the portfolio involves selling part of it and a possible purchase of another bond so immunization is maintained. This process is both costly and risky. The commissions involved cause the process to become costly and it is risky because the bonds which need to be sold may not be easily marketable, bonds which need to be purchased may not be readily available, or the risk of default may be too great for the desired bonds.

In reality, the yield curve is constantly changing its shape and its level. Therefore, any stochastic process assumed will probably not correctly describe the actual one. Perfect immunization will not be realized. In other words, the actual yield will differ from the promised yield. Clustering the individual durations around the length of the holding period helps to reduce possible losses. This may not always be possible with the current supply of securities.

4. Immunization in practice

Firms do actually employ these immunization strategies. Smith Barney offers their clients three immunization products. Of the 30 portfolios they manage currently using one of these products, over 90 percent are earning a return greater than the target rate. Of the ten percent earning a lower rate of return, none is more than 17 basis points below the targeted return. Smith Barney was the first company to apply the concept of immunization to the management of pension funds.

Another company currently offering immunization products to their clients is Trust Company of the West. They noticed that due to the poor performance of the stock market in 1969-1970 and 1973-1974, firms again began to consider bonds as investments superior to stocks. During the 1975-1977 period, their portfolio managers sold stocks and purchased bonds. Active management strategies became more prevalent. Today, Trust Company of the West is using financial futures to alter the durations of their actively managed portfolios. Simulations indicate this strategy looks very promising. Trust Company feels when fund managers and sponsors become more familiar with financial futures and active management strategies, more firms will begin to offer such services.

Manufactures Hanover Trust currently manages over $1 billion using immunization strategies. They ran simulations from 1958 through 1979 and found immunization is consistently achievable. Their newest strategy is called an "active risk-controlled management system" (ARMS) but it is nothing more than a contingent immunization strategy.

Large firms in the busines of managing portfolios do actually use immunization techniques. All of the ones mentioned here have had a great deal of success. These firms are continually developing new products to fit the needs of specific clients. In the future, a continuation of the intense interest shown in the past seems likely and within a short time financial futures and financial futures options will be used to a larger extent. This area of active bond management is relatively new but should be a useful management tool.

IV. LITERATURE REVIEW

A. Financial Futures

The amount of literature relating to the financial futures market is relatively sparse because the first contract was not traded until 1975. Tests of market efficiency were not published until late in 1978 and early in 1979. The uses of the U.S. Treasury Bill (T-bill) and Government National Mortgage Association (GNMA) market for hedging purposes were first described by Bacon and Williams (2).

The T-bill futures contract stipulates the delivery of $1,000,000 face value of 90-, 91-, or 92-day U.S. Treasury Bills. Contract prices are quoted assuming delivery of 90-day T-bills so an adjustment to the invoice amount is necessary if the 91- or 92-day T-bills are delivered in fullfilment of the contract. The GNMA contract stipulates delivery of $100,000 principal amount of an eight percent coupon of GNMA certificates representing shares in pools of Veterans Administration or Federal Housing Administration mortgage loans. These loans have a maturity of 30 years with prepayment expected at the end of 12 years.

Bacon and Williams (2) showed how to achieve a perfect cross hedge by selling GNMA contracts to hedge the issuance of corporate bonds provided the coupon rates, maturities, and yield changes were the same for both the GNMA futures contract and the corporate bonds. Later they relax the assumption of equal coupon rates and equal maturities to show that in practice hedging will normally not be perfect. They also provide examples of cross hedging against rising and falling short-term interest rates using T-bill futures contracts as the hedging vehicle.

Ederington (12) was the first person to use a basic portfolio model as a means of measuring the hedging performance of the financial futures market. The formula he used to calculate the risk minimizing hedge ratio is presented again.

$$(\Delta Ps) = a + b^*(\Delta Pf). \tag{9}$$

A regression is run using time series data to estimate the hedge ratio, b^*, where ΔPs equals the change in the spot price and ΔPf is the change in the futures price where both changes are from period t to period t+k.

Ederington also defined a measure of hedging effectiveness as the percent reduction in the variance

$$e = 1 - \frac{\mathrm{var}(R^*)}{\mathrm{var}(U)} \tag{16}$$

where $\mathrm{var}(R^*)$ is the minimized variance for the return of a portfolio containing spot and futures holdings and $\mathrm{var}(U)$ is the variance of an unhedged position. An estimate of e is provided by the sample coefficient of determination, r^2, between the change in prices of the spot instrument and futures contract.

Assuming a hedging period of two and four weeks, he tested the GNMA and T-bill futures markets for hedging effectiveness. Hedging effectiveness is measured by the reduction in the variance of returns on a portfolio containing futures contracts and cash instruments. His results indicate both e and b* are larger for the four-week hedging period. Wheat and corn futures were compared to GNMAs and T-bills for hedging a cash wheat or corn position and it was found the agricultural futures contracts had a much higher estimated e (reduction in the variance of returns) and a b* closer to 1.0.

Franckle (14) described two flaws in Ederington's model. The first is that Ederington did not take into consideration the fact that as a T-bill approaches maturity its price will change. Secondly, the price senitivity of the cash T-bill will decrease as it approaches maturity. The correction for the first problem involves using the proper variance and covariance. Franckle argues that Ederington should have used the difference of a 90-day discount rate and a 76- or 62-day discount rate instead of another 90-day discount rate. The adjustment to the covariance would include the above plus the change in the discount rate for the futures contract. Franckle adjusted the hedge ratio to account for the decreasing maturity of a T-bill by assuming a 90-day bill at the beginning of the hedging period but only a 76- or 62-day bill at the end. His results show these two corrections decrease the value of b* but do not substantially alter the value of e for the four-week hedge. The results for the two-week period using T-bills indicate the effectiveness (e) of using this short hedging period is increased.

Hill and Schneeweis (18) authored one of the first articles describing the crosshedging usefulness of T-bond futures. Their model assumed a spot position in corporate bonds (as measured by an index such as Moody's Corporate AAA) and a futures position in T-bonds or GNMAs. Using Ederington's model they found hedging could reduce the variance of value changes for the bond indices by as much as 50 to 90 percent. The high quality corporates and public utility bonds occupied the higher end of this range. They also found that hedging with the nearby contract reduced the variance of monthly price changes more than the distant contracts. The T-bond contract was consistently the more effective hedging vehicle.

Another method used for calculating the optimal hedge ratio, which incorporates the use of duration, was developed by Kolb and Chaing (21). Called the price sensitivity (PS) model, its goal is to choose some number of futures contracts to trade so that the change in the price of the asset plus the product of the change in the price of the futures contract and the number of contracts traded equals zero. This has been described earlier in equation (5). The general solution for the number of contracts to trade, N, was given in equation (3). This method is superior to the others because it considers the coupon, risk level, and maturity of the hedged and hedging instruments. The assets' price sensitivities to changes in interest rates are acknowledged by using their durations. The authors point out that Franckle's adjustments to Ederington's model are not needed in their model. They also discuss the usefulness of this method for institutions that need a cross-hedge.

The general model used in this study is patterned after the model developed by Chance (9). His model involves an initial hedging period prior to the actual purchase of the portfolio (i.e., anticipatory hedge) and an immunization strategy for the entire length of the holding period. This model is then compared to a model which only uses hedging as a means of protecting the value of the portfolio. He does not run simulations on actual data so a statement regarding the usefulness of the model is unavailable.

B. Duration-Immunization Theory

While conducting a study of bond prices and interest rates in 1938, Macaulay (25) realized that term to maturity did not provide the investor with an accurate description of how a bond's price will react to changing interest rates. Consequently, he derived equation (1). This equation weights the present value of the coupon payments as well as the present value of the payment at maturity. The weights are the present values of all the cash flows as a percentage of the total present values. Macaulay called this measure duration.

Hicks (17), shortly after Macaulay, also developed equation (1). He calculated the elasticity of a discounted series of cash flows, where the discount factor was (1 + i). This elasticity is measured in units of time (years) and he called it the "average period." Neither Hicks or Macaulay demonstrated the usefulness of duration, consequently it never received a great deal of attention.

The man responsible for the concept of immunization was a British actuarist, F. M. Redington (28). While studying the effects of unexpected changes in interest rates on the profits of a life insurance company, he discovered the "mean term" (first derivative) of the values of the cash flows with respect to interest rates determined the manner in which assets should be invested to immunize the company from changing interest rates. When the first derivative of the cash inflows is equal to the first derivative of the cash outflows, the possibility of suffering a loss is minimized. This type of immunization is applicable to financial intermediaries and is not the focus of this study.

Fisher and Weil (13) demonstrated the applicaton of duration and immunization to the academic community. They showed that when the duration of a bond portfolio was equated to the length of an investor's holding period, changes in interest rates would not have an adverse affect on the value of the portfolio. Unlike equation (1) where the discount factor is the same for each period (flat term structure), their formula allows for changes in the one-period discount rates. Like Macaulay, they assumed all rates are perfectly correlated. This constrains the model to allow only for parallel shifts in the yield curve.

Hopewell and Kaufman (20) specified a theorem of bond price volatility. They noticed that analysts during this time did not thoroughly understand the mathematics of bond pricing. A simple relationship between the volatility of a bond's price and its term to maturity did not exist. By taking the differential of the price of a bond with respect to the interest rate, they found a simple relationship between the duration of a bond and its price volatility. Equations (11a) and (11b) show that as the duration of a security increases, there is a proportional percentage change in the price of a bond for a given change in the discount rate. They also incorrectly proposed that yield curves should be derived using duration instead of time to maturity.

The accuracy of immunization procedure developed by Fisher and Weil depends upon the assumed random shock to the term structure. Bierwag and Kaufman (5) proposed a measure of duration which did not embody the assumption that changes in all interest rates have to be perfectly correlated with one another. The formulas for calculating duration developed by Macaulay and Fisher and Weil assumed an additive shock to the term structure. Adding a bit of realism, they formulated a measure of duration which assumes the random shock is multiplicative in nature. An example of this type of shock would be for short-term rates to be affected more than long-term rates.

A test was performed to compare different measures of duration (Macaulay's, Fisher and Weil's, and Bierwag and Kaufman's) assuming various terms to maturity, yields, and coupons. These measures were discussed in Chapter III, Section C, and were called D1, D2, and D3, respectively. It was found that the three different underlying stochastic processes generating changes in interest rates produced values of duration very close to one another. The authors point out that since Macaulay's measure is a function of the yield to maturity of the bond, it becomes unnecessary to forecast the one-period forward rates.

McEnally (26) lucidly explained the more pragmatic aspects of duration. He showed how to calculate the duration of a coupon bond using present value tables and also presented a few examples to demonstrate the usefulness of duration for portfolio managers. McEnally explained that although one portfolio may have a longer duration than another, the differences in yields between the two may justify investing in the one with the longer duration. Finally, he showed that duration has a special feature which he calls additivity. This means a group of securities whose duration is equal to ten years will behave exactly as another portfolio whose duration is also ten years. This is not the case for the weighted average term to maturity.

Bierwag (4) claimed duration is nothing more than an index number which implies that all the information contained in a series of cash flows cannot be incorporated into this simple formula. He did recommend that if an investor's objective is immunization, duration-matching should be used. The difficult part is calculating the weights to be used in the index because they are a function of the investor's subjective view as to how the term structure will change. Since this is never known with certainty, more often than not the incorrect measure wil be applied. As a result, the investor should not expect an immunization strategy to perform perfectly.

One criticism of the measures of duration developed by the previous authors is that the measures only apply to a one-time shift in the term structure. Cox, Ingersoll, and Ross (11) describe a formula for duration which is able to account for multiple shocks to the term structure. This measure may be used for a yield curve with a changing shape. They compared values for their measure of duration versus Macaulay's and Fisher and Weil's measures. Their values are always lower than the more simple measures. This implies the other measures overestimate the riskiness of a portfolio. A problem with this dynamic method is that it is very complex. It has its roots in operations research which typically tries to solve dynamic problems by reducing them to a static problem. No current studies which have been published use this method even though the authors claim it is theorectically more correct.

Yawitz and Marshall (32) concluded duration cannot be used as a linear measure of risk. They show it may be used as an ordinal risk measure as may term to maturity. The only way for duration to be considered as a linear risk measure is to assume when yields change, they all change by the same amount (i.e., parallel shift in the yield curve). They concluded that duration does correctly account for differences in coupons and maturities but it cannot solely be used as a measure of bond price volatility.

Duration and immunization research has increased in the past few years. The academic community continues to search for a measure of duration which is theoretically correct and applicable. Toward this end, a new type of immunization, contingent immunization, has recently been proposed and has been accepted by the money management industry.

V. DEFINING AND TESTING THE MODEL

A. Definition of the Model

Chapter I defined the problem; Chapters II and III discussed duration, immunization theory, hedging methods, general aspects of the commodities futures market, and specific features of the financial futures market. This chapter will combine the theory of Chapters II and III for the purposes of formulating a hedging-immunization investment model. This model is designed to provide the portfolio manager with a close approximation of the future value of the portfolio regardles of how interest rates change during the holding period. Figure 5 is a schematic of the model.

t0	t1	tn
Decide to purchase portfolio	Purchase portfolio	Liquidate portfolio

Hedging period (t0 to t1)	Immunization period (t1 to tn)

Figure 5. Hedging-immunization model schematic

The decision to purchase a portfolio of AAA corporate bonds is made at t0, but the actual purchase is not made until t1. Interest rates may fall during this period causing the market price of the bonds to increase. A correctly formulated hedging strategy will result in a gain (loss) on the futures transaction which exactly offsets the opportunity loss (gain) caused by changing bond prices between t0 and t1. The hedging strategy will be formulated using the price sensitivity (PS) model tested earlier.

The immunization procedure will allow the manager to realize the promised yield available in the current term structure at t1. This is accomplished by setting the duration of the portfolio equal the length of the investor's holding period. Regardless of the direction interest rates change during the holding period, immunization assures the investor that the change in income due to reinvesting the semiannual coupon payments will be equal, but opposite, to the change in the market value of the portfolio. Suppose a manager knows he/she has a fixed liability payment in the future and is going to fund this payment by investing in high grade corporate bonds. Since the manager is able to closely approximate the future value of the portfolio, he/she is able to determine the size of the initial investment necessary to satisfy the liability. A pension fund manager may find this feature very attractive.

Unlike the model developed by Chance (9), this model does not take a simplistic view of the hedging process. Chance determines the hedge ratio using the naive model which is inefficient. The naive model has been shown to be far too inconsistent in the sense that a perfect hedge is usually not even remotely achieved. Refer to Table 3 for a comparison of the various hedging models. This model adjusts the duration of the portfolio after each semiannual coupon payment so immunization is maintained whereas Chance does not readjust his portfolio to account for its decreasing duration.

The model assumes a hedging period of two months and a holding period of two years. The T-bond futures contracts are purchased on December 3, 1979 and this position will be offset by a sale of T-bond futures contracts on February 1, 1980. The $1 million bond portfolio consists of AAA rated American Telephone and Telegraph (ATT) 7 3/4s of 1982 and AAA General Motors Acceptance (GMA) 8 1/8s of 1984. The holding period spans from February 1, 1980 until February 2, 1982. The target or promised rate of return for this holding period was determined to be 12.35 percent. This was the weighted average return for the futures contract during the hedging period and the two-year bond for the immunization period. This target was chosen because the hedge would have locked in a 10.059 percent yield for two months and the immunization strategy a 12.5413 percent return for two years.

B. Testing the Model

This section will describe the hedging strategy, the portfolio adjustment process, and the results for the investment strategy discussed in Section A. Figure 6 lists all bond prices, yields, and durations for each point in time when the portfolio is readjusted (i.e., after each semiannual coupon payment). It also lists the T-bond futures contract prices, yields, and duration used for hedging. The readjustment procedure is necessary to keep the portfolio immunized. Both of the bonds mature in February and have coupon payments in the month of

MARCH T-BOND FUTURES CONTRACT

Date	Price	Yield	Duration
12/3/79	82-13	10.059	9.359
2/1/80	74-16	11.226	-

AAA CORPORATE BONDS

	ATT 7 3/4 '82			GMA 8 1/8 '84		
Date	Price	Yield	Duration	Price	Yield	Duration
12/3/79	93 1/2	11.15688	2.066	88 1/4	11.71596	3.575
2/1/80	91 3/4	12.54130	1.885	87 1/2	12.16398	3.453
8/2/80	97 1/4	8.76520	1.443	92 3/4	10.66098	3.099
2/3/81	94 1/4	14.11566	0.981	86 1/4	13.88406	2.697
8/1/81	95 3/4	17.02794	0.500	83	16.72166	2.291
2/2/82	99 23/32	-	-	88	-	-

Figure 6. Simulation price, yield, and duration data

maturity and also in August. This reduces the number of times the portfolio will have to be readjusted by half because the duration of the portfolio changes after each coupon payment.

The first step is to calculate the number of futures contracts to buy using the PS method. Equation (3) gives the general form to use when calculating the hedge ratio.

$$N = \frac{\overline{R}_j \; P_i \; D_i}{\overline{R}_i \; FP_j \; D_j} \tag{3}$$

where $\overline{R}_j$ is equal to 1 + the yield for the futures contract, P_i is the price of asset i, D_i is the duration of asset i, $\overline{R}_i$ equals 1 + the yield of asset i, FP_i is the price of the futures contract, and D_j is the duration of the futures contract. The hedge ratio, N, is calculated for each bond (0.00247994 for the ATT bond and 0.00403006 for the GMA) and is then multiplied by the number of each type of bond purchased (998 ATTs and 76 GMAs). This implies the purchase of 3.099 futures contracts. The hedger will purchase three March T-bond futures contracts because fractions of contracts are not traded. Notice that this is a slightly bearish position. The March contract was chosen because it has the largest trading volume and open interest (i.e., liquidity).

The price of the T-bond futures contract declined from 82-13 to 74-16 during the hedging period. This resulted in a loss of $7,906.25 per contract. Since three contracts were purchased, the total futures loss was $23,718.75. An unrealized gain on the cash side helped to offset most of the futures loss. The prices of the ATT and GMA bonds decreased

$17.50 and $7.50, respectively. Multiplying these opportunity gains by the number of each type of bond purchased (998 ATTs and 76 GMAs) resulted in an unrealized gain of $18,035. A summary of the hedge is presented in Table 8.

Table 8. Hedge results

Futures loss	$23,718.75
Bond Gain (unrealized)	18,035.00
Net Loss	$ 5,683.75

It should be understood that part of the decline in the prices of the corporates could have been caused by riding up a downward-sloping yield curve. An examination of yield curves for Treasury issues on November 30, December 31, and January 31 showed that the yield curve was indeed downward-sloping. More important was the rise in the level of the yield curve over this two-month period and its general flattening-out. Based on these observations, the majority of the increase in bond prices for this period was probably due to the shifting of the yield curve and not due to riding the yield curve.

The simulation has shown hedging results in a loss for the investor. In some cases this loss could have been reduced by the use of a special type of market order called a stop-loss order. Here is how it could have been used. The goal of purchasing the T-bond futures contracts was to protect the investor from large opportunity losses on the cash side caused by falling interest rates. Falling interest rates would have been profitable for a long futures position. As it turned out, rates rose over the hedging period. The investor could have placed a stop-loss order at 78-00 which would have instructed his/her broker to offset the long position if T-bond futures prices fell to this level. This would have decreased the hedging loss from \$5,683.75 to a net gain of \$4,816.25.

The art of using a stop-loss order is knowing where to position it. It must be placed far enough away from the original price so that if it is reached the manager can be relatively sure a trend is being established. Another possibility is to evaluate the market as it approaches the stop-loss price and change it if it is felt the market is reversing its trend. For example, suppose the order is placed at 78-00 but the market begins to turn bullish. If the futures price suddenly increases to 81-00, the stop-loss price could be moved to 80-00 to protect part of the gain. This whole process is complicated and borders on speculation.

On February 1, 1980, the duration of the portfolio is set equal to two years (i.e., the length of the holding period). A simple weighted average equation is used

$$2.0 = Wa(D_{ATT}) + (1-Wa)(D_{GMA}) \tag{17}$$

where Wa is the percentage of the \$1 million invested in the ATT bond, (1-Wa) is the percentage invested in the GMA bond, D_{ATT} is the duration of the ATT bond on February 1, 1980, and D_{GMA} is the duration of the GMA bond on this date also. Wa is equal to 0.927 which implies an investment of \$927,000 in the ATT 7 3/4 of 1982 and \$73,000 in the GMA 8 1/8 of 1984. The initial purchase will be 1,010 (\$927,000/\$917.50) ATTs and 83 (\$73,000/\$875.00) GMAs.

The first semiannual coupon payment is received on August 2, 1980. The 7 3/4 ATT bonds provide \$39,137.50 (1,010 bonds · \$38.75) coupon income and the 8 1/3 GMA bonds provide \$3,371.88 (83 bonds · \$40.625) for a total of \$42,509.38. Only 1.5 years remains in the holding period so equation (17) is used to calculate the percentage of each bond comprising the portfolio. The ATT bond now accounts for 96.6 percent of the portfolio and the GMA bond only 3.4 percent. Since the ATT (GMA) originally comprised 92.7 (7.3) percent of the portfolio, \$39,000 (96.6% - 92.7%) must be used to purchase additional ATT bonds. The \$42,509.38 of coupon income will be invested in both bonds according to their revised percentages. The net adjustment results in the purchase of 82 ATTS and the sale of 40 GMAs. Figure 7 presents a detailed view of the semiannual portfolio adjustment process. This procedure is the same for each six-month period so only one duration adjustment is presented.

By the end of the two-year holding period, the portfolio does not contain any GMA bonds. This is caused by the behavior of duration for bonds sold at a discount. Recall in Figure 3 it was shown duration increases (for this example decreases) monotonically for these types of bonds. Consequently, while maturity (duration) is decreasing, the duration of the shorter-term bond will decrease more rapidly than the duration of the longer-term bond. At the end of the third coupon period (1.5 years), the duration of the ATT bond has fallen to 0.50. This is

DATE: AUGUST 2, 1980

Coupon income:

7 3/4 ATT	1,010($38.75) =	$39,137.50
8 1/8 GMA	83($40.625) =	3,371.88
		$42,509.38

Duration adjustment:

$$Wa(1.443) + (1-Wa)(3.099) = 1.5$$
$$Wa = 0.966$$

% of portfolio comprised of ATTs: 96.6
% of portfolio comprised of GMAs: 3.4

Originally Wa = 0.927 which means the investor now must sell $39,000 of the GMAs and purchase $39,000 of ATTs.

$39,000/$972.50 (current price ATT) = 40 bonds purchased
$39,000/$927.50 (current price GMA) = 42 bonds sold

Reinvestment of coupon income:

		Bonds purchased
0.966 ($42,509.38) = $41,064.06;	$41,064.06/$972.50 =	42 ATTs
0.034 ($42,509.38) = 1,445.32;	1,445.32/$927.50 =	2 GMAs

Net adjustment: Purchase 82 ATTs and sell 40 GMAs.

Figure 7. Portfolio adjustment process for August 2, 1980

equal to the time remaining in the holding period. The investor must sell all of the GMAs in the portfolio or else its duration will be too large.

The long bond position consisting of 1,227 7 3/4 ATTs of 1982 is liquidated on February 2, 1982. The coupon income for this period is \$47,546.25 (1,227(\$38.75)). The proceeds from the sale totaled \$1,223,549.10. The coupon income and the sale proceeds add up to \$1,271,095.30 or a yield of 12.3609 percent for the two year holding period. The hedging loss must be considered also. The \$5683.75 loss would have had a future value of \$7,224.59 compounded semiannually at 12.3609 percent. This reduces the yield to 12.0587 percent or by 30.22 basis points. Recall that the target rate of return is 12.35 percent. This simulation shows the realized return differs from the promised return by 1.09 basis point without considering the hedge loss or by 29.13 basis points when the hedge loss is considered.

The investor should be aware of the transaction costs when using this strategy. Round-trip commissions in the T-bond futures market are roughly \$50 per contract. Bond commissions vary with the number of bonds traded. The investor should expect to pay \$5 per bond if the number of bonds traded is greater than 100. If the number traded is less than 100, the commission will consist of a flat fee plus a certain amount for each bond. The total transaction costs for the immunization technique, using the rates quoted by a full-service broker, totaled seven-tenths of one percent of the initial investment. Rates charged to a regular customer or by a discount broker are probably substantially less than those listed above.

VI. SUMMARY AND CONCLUSIONS

The purpose of this study was to define and test an investment strategy which reduces the investor's exposure to interest rate risk. A major problem encountered when managing a portfolio of fixed income securities is the fluctuation of the portfolio's market value caused by varying interest rates. The reinvestment rates for the semiannual coupon payments are also subject to these changing rates. Along with this is the added risk that during the time period prior to the actual investment, interest rates may fall resulting in higher market prices for the securities. These two types of risk, price risk and reinvestment risk, may be managed effectively using a hedging-immunization strategy.

The commodity futures market is an open-outcry market. Prices are determined by traders shouting bid and asked prices in the trading pits at the exchanges. Participants enter the market either to speculate or to hedge. Speculators normally do not have a use for the commodity they trade nor do they hold a cash position in it. Arbitrageurs search the market for price imbalances and try to profit from these anomalies. They help to keep the market efficient. Spreaders also watch for price imbalances but they assume a level of risk somewhere between the speculator and the arbitrageur. Speculators add liquidity to the market which allows for easy entrance and exit. Hedgers, unlike speculators, usually own or use the commodity they trade. Their purpose for entering the market is to lock-in an acceptable price for the commodity they are planning to purchase or sell. Speculators are profit maximizers but hedgers are risk reducers.

An attractive feature for both the speculator and the hedger is the amount of leverage one is able to obtain by trading T-bond futures contracts. An initial margin of \$1,250 enables the participant to assume a position in a contract whose underlying assets are worth \$100,000. A maintenance margin is also required which has just been reduced to \$1,000. This feature makes the futures market much more accessible to speculators and hedgers.

A futures contract specifies the characteristics of the asset which is deliverable against the contract. It also specifies the delivery date(s) and outlines the three-day delivery process. Each contract must have a buyer and a seller but the two parties never know with whom they are trading. Orders are sent to brokers who relay the imformation to the trading floor. The brokers deal with a clearing member who, in turn, deals with the clearinghouse. The clearinghouse is authorized to daily mark-to-market each trader's account. The number of long and short positions are equal so the net position of all the buyers and sellers is zero everyday.

The futures contract used for hedging purposes in this study was the T-bond contract. It calls for the delivery of an eight percent Treasury bond with at least 15 years remaining before the maturity or call date. Many issues qualify for delivery at a point in time so a factor system was developed to price the Treasury issues which do not have an eight percent coupon and exactly 15 years remaining to maturity or call. The factor will be larger than one for an issue which has a coupon greater than eight percent and less than one if the coupon is less than eight percent. Since there are numerous Treasury issues which qualify for delivery, the shorts will try to deliver the cheapest bond available. Normally the newest issue with the highest coupon will be the cheapest to deliver.

The hedger must determine the optimal number of contracts to trade for his/her specific situation. This is not an easy process due to the multitude of factors which may affect the hedge ratio. These include the shape of the yield curve, the market value of the cash position relative to the market value of the futures contract, price correlation between the cash and futures instrument, the maturity of the asset underlying the futures contract and the maturity of the cash instrument, the coupon rate for the futures contract and the cash instrument, and the differences in the risk structure of interest rates. Some of the factors are known prior to implementing the hedge but the others will have to be estimated. The PS model simplifies estimating these variables by assuming a flat yield curve. Hedging is advantageous because the participant reduces his/her risk from the extremely volatile price risk to the more manageable basis risk.

An anticipatory hedging possibility provided by the T-bond futures market was examined. The goal of hedging was to keep the hedger's initial position intact. In other words, the opportunity gain (loss) on the cash side should equal the total loss (gain) from the futures transaction. Four models were tested and it was found that the PS model consistently provided the hedger with the closest approximaton to a perfect hedge. It was pointed out that the objective of the portfolio theory model is not the same as the objective of the PS model.

The second part of the investment strategy centered around the concept of duration. Discovered over 40 years ago, it has not received much attention until very recently. Duration is a weighted average time to maturity where the weights are expressed in present value terms. The duration of a coupon bond is always less than its time to maturity but the duration of a zero coupon bond is equal to its time to maturity. The behavior of duration for a bond sold at a discount is different from the bahavior of a bond sold at a premium or at par. In summary, duration is a function of the coupon, yield, and maturity of a bond where the first two arguments have a negative affect upon duration and the last argument a positive affect.

There are many different measures of duration. Each measure embodies an assumption concerning the underlying stochastic process generating changes in interest rates. This study discussed three different measures. It was found that although Macaulay's (25) simple measure contains an unrealistic assumption concerning the shape of the yield curve, it does perform reasonably well in practice. Complex measures which are theoretically more pleasing do not perform well in practice and are not widely used in the literature.

A portfolio is immunized if its duration is set equal to the length of the investor's holding period. This is known in the literature as classical immunization. It is desirable because the investor is given a close approximation for the future value of his/her investment regardless of how interest rates change during the holding period. A portfolio manager may wish to try to increase the portfolio's return by taking a more active approach. This entails setting the duration of the portfolio either less than or greater than the holding period. If the manager feels interest rates will decline in the future he/she should set the duration greater than the length of the holding period. This approach is called active management whereas the classical approach is known as passive management.

An investor becomes exposed to stochastic process risk when the assumed underlying stochastic process generating changes in interest rates does not coincide with the actual process. This type of risk will usually be present because the measures of duration assume a one-time shift in the term structure whereas shape of the yield curve is continually changing. One way to reduce this risk is to condense the individual durations in the portfolio around the length of the holding period. This causes the portfolio to behave more closely to an immunized portfolio comprised of zero coupon bonds. The zero coupon bond portfolio is always immunized regardless of the actual stochastic generating process.

Duration theory contains a few theoretical flaws and practical problems. The measures discusssed in this paper will provide the investor with a greater realized return than promised simply because interest rates change assuming the portfolio is immunized. Another problem is caused by the fact that a riskless arbitrage opportunity is possible simply by switching from a portfolio containing low coupon bonds to one containing higher coupon bonds of equal duration. The measures presented in this study are called single-factor duration models because only one factor is needed to describe the interest rate generating process. The duration problem greatly increases in complexity if the underlying stochastic process cannot be completely described by a single factor. Finally, the duration of a portfolio containing coupon bearing securities will not decrease at the same rate as the length of the holding period. This implies that the portfolio will have to be adjusted periodically to maintain immunization.

A. Conclusions

Hedging has been shown to result in a net loss for the investor. This does not mean that hedging should be avoided. It does show that the most unpredictable part of this investment strategy is the hedging process. As noted earlier, the manager should monitor the market closely to determine general interest rate trends and he/she should also consider using stop-loss orders. A stop-loss order placed at 78-00 would have resulted in a net gain of $4,816.25 for the hedge.

Another problem with using this strategy is that the futures contact price may not always move in the same direction as all the cash bond prices. On a different simulation, the market price of the shorter-term bond increased during the hedging period, the market price of the longer-term bond decreased, and the futures price decreased. Since most of the portfolio was comprised of the shorter term bond, the investor suffered a loss on the long futures position and on the cash side. This was probably due to a shifting of the yield curve from downward-sloping configuration to a humped shape.

The immunization procedure was shown to perform reasonably well. The realized return differed from the promised retun by only 1.09 basis points. This is similar to the results shown earlier in Table 7. The reason why the realized return is not equal to the promised return is because the assumed underlying stochastic process generating changes in interest rates did not correspond to the actual stochastic process. The investment suffered from stochastic process risk.

This paper has presented a hedging-immunization investment strategy. It does work reasonably well in practice but problems do develop. Hedging seems to be the most unpredictable part of this technique. The immunization strategy has been shown to realize a rate of return slightly lower than its promised rate. The portfolio manager may decide to take a more "active" approach by using stop-loss orders or predicting changing interest rate trends. The profit potential is increased for this approach but then so is the level of risk. Each manager must decide for himself/herself which approach is the prudent one to adopt.

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